J. Myanmar Acad. Arts Sci. 2018 **Vol. XVI. No.3**

DECOMPOSITION OF COMPLEX VECTOR SPACE C^n INTO INVARIANT SUBSPACES

Myint Myint Maw*

Abstract

This paper study the existence of eigenvalue for every linear operator on a finite-dimensional complex vector space. In this paper, we will discuss although eigenvectors corresponding to distinct eigenvalues are linearly independent, they can not span the complex vector space. Then we give decomposition of complex vector space C^n into generalized eigenspaces and Jordan subspaces.

Keywords: Invariant subspace, Jordan chain, Generalized eigenspace, Jordan subspace

1. Eigenvalues and eigenvectors

Throughout the paper, V denotes n-dimensional complex vector space.

1.1 Definition. Let $A : V \to V$ be a linear operator. A subspace $M \subset V$ is called ***invariant*** for the linear operator A, or A-invariant, if $Ax \in M$ for every vector $x \in M$.

Trivial examples of invariant subspaces are $\{0\}, V$, Ker $A = \{x \in V \mid Ax = 0\}$ and
Im $A = \{Ax \mid x \in V\}$.

1.2 Definition. Let A: $V \to V$ be a linear operator. A number $\lambda \in C$ is called an ***eigenvalue*** of A if there exists $x \in V$ such that $x \neq 0$ and $Ax = \lambda x$. The vector x is called an ***eigenvector*** of A corresponding to λ.

1.3 Theorem. Let $A : V \to V$ be a linear operator and $\lambda \in C$. Then the following are equivalent:

(a) λ is an eigenvalue of A.

(b) $A - \lambda I$ is not injective.

* Lecturer, Department of Mathematics, Dagon University

(c) $A - \lambda I$ is not surjective.

(d) $A - \lambda I$ is not invertible.

Proof. λ is an eigenvalue of A $\Leftrightarrow \exists v \in V$ such that $v \neq 0$ and $Av = \lambda v$.

$\Leftrightarrow (A - \lambda I)\, v = 0$

$\Leftrightarrow A - \lambda I$ is not injective.

Thus conditions (a) and (b) are equivalent.

Clearly conditions (b), (c) and (d) are equivalent.

1.4 Theorem. Every linear operator on a finite-dimensional complex vector space has an eigenvalue.

Proof. To show that A has an eigenvalvue, choose a non-zero vector $v \in V$. We consider the $n + 1$ vectors $v, Av, A^2v, \ldots, A^n v$. Since the dimension of V is n, $v, Av, A^2v, \ldots, A^n v$ are not linearly independent.

Thus there exist complex numbers $a_0, a_1, \ldots, a_n$, not all zero such that $a_0 v + a_1 v + \ldots + a_n A^n v = 0$.

Make the a's the coefficients of a polynomial, by the Fundamental Theorem of Linear Algebra which can be written in factored form as

$$a_0 + a_1 z + \ldots + a_n z^n = c\,(z - \gamma_1) \ldots (z - \gamma_m),\ m \leq n$$

where$m$ is largest positive integer such that $a_m \neq 0$, c is a non-zero complex number, each γ_j is complex and equation holds for all complex z. We then have

$$\begin{aligned} a_0 v + a_i A v + \ldots + a_n A^n v &= 0 \\ (a_0 I + a_1 A + \ldots + a_n A^n) v &= 0 \\ (c(A - \gamma_1 I) \ldots (A - \gamma_m I))\, v &= 0. \end{aligned}$$

We know that the composition of injective mappings is injective and $v \neq 0$. Thus $A - \gamma_j I$ is not injective for at least one j. In other words, A has an eigenvalue.

1.5 Proposition. Non-zero eigenvectors corresponding to distinct eigenvalues of A are linearly independent.

Proof. Suppose that $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of A and $v_1, \ldots, v_m$ are corresponding non-zero eigenvectors. We need to prove that $v_1, \ldots, v_m$ are linearly independent. Suppose that $a_1, \ldots, a_m$ are complex numbers such that $a_1v_1 + \ldots + a_mv_m = 0$. Apply the linear operator $(A - \lambda_2 I)(A - \lambda_3 I) \ldots (A - \lambda_m I)$ to both sides of the equation above,

$$((A - \lambda_2 I)(A - \lambda_m I) \ldots (A - \lambda_m I)(a_1v_1 + \ldots + a_mv_m) = 0.$$

Since we have $(A - \lambda_j I)v_j = 0$, $j = 1,2, \ldots, m$ and two polynomials in the same linear operator are commute, then we have

$$((A - \lambda_2 I)(A - \lambda_3 I) \ldots (A - \lambda_m I)(a_1v_1) = 0.$$

But $(A - \lambda_j I)v_1 = Av_1 - \lambda_j (Iv_1) = \lambda_1 v_1 - \lambda_j v_1 = (\lambda_1 - \lambda_j)v_1$ for $j \neq 1$.

Thus $a_1 (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3) \ldots (\lambda_1 - \lambda_m) v_1 = 0$. Since λ's are distinct eigenvalues and v_1 is non-zero eigenvector, we get $a_1 = 0$. In a similar fashion, $a_j = 0$ for each j.

1.6 Definition. Suppose $A : V \to V$ and $\lambda \in \boldsymbol{C}$. The ***eigenspace*** of A corresponding to λ, denote by $E(\lambda, A)$, is defined by $E(\lambda, A) = \text{Ker}\,(A - \lambda I)$.

1.7 Theorem. Suppose V is finite-dimensional and $A: V \to V$. Suppose also that $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of A. Then $E(\lambda_1, A) + \ldots + E(\lambda_m, A)$ is a direct sum and $\dim E(\lambda_1, A) + \ldots + \dim E(\lambda_m, A) \leq \dim V$.

Proof. We know that the null space of each linear mapping on V is a subspace of V.

Thus $E(\lambda_1, A) + \ldots + E(\lambda_m, A)$ is a subspace of V.

Take any $x \in E(\lambda_1, A) \cap E(\lambda_j, A)$ for $i \neq j$.

So $(A - \lambda_i I)x = 0$ and $(A - \lambda_j I)x = 0$.

$Ax = \lambda_i x$ and $Ax = \lambda_j x$ this implies that $\lambda_i x = \lambda_j x$ so $(\lambda_i - \lambda_j)x = 0$.

Since λ's are different, we get $x = 0$. Thus $E(\lambda_1, A) + \ldots + E(\lambda_m, A)$ is a direct sum of V.

Hence $\dim(E(\lambda_1, A)+\ldots+E(\lambda_m, A)) = \dim E(\lambda_1, A) +\ldots+ \dim E(\lambda_m,A) \leq \dim V$.

1.8 Remark. Non-zero eigenvectors corresponding to distinct eigenvalues of A need not span V.

1.9 Example. The linear operator $A : C^2 \to C^2$ defined by $A(w, z) = (z, 0)$.

$\forall\ (w, z) \neq (0, 0)$ and $\forall \lambda \neq 0$ in C, $\lambda\,(w, z) \neq (z, 0)$.

Thus to get $A(w, z) = \lambda\,(w, z)$, $\lambda = 0$ is forced, and so 0 is only eigenvalue of A. The set of eigenvectors corresponding 0 is $\{(w, 0) \in \mathbf{C}^2) \mid w \in \mathbf{C}\}$ it is one dimensional subspace of $\mathbf{C}^2$. Clearly $(w, 0)$ cannot span $\mathbf{C}^2$.

2. Generalized Eigenspaces

2.1 Definition. Let λ be an eigenvalue of a linear operator $A : C^n \to C^n$. A chain of vectors $x_0, x_1, \ldots, x_k$ is called ***Jordan chain*** of A corresponding to λ if $x_0 \neq 0$ and the following relation hold:

$$
(1) \qquad \begin{aligned}
Ax_0 &= \lambda x_0 \\
Ax_1 - \lambda x_1 &= x_0 \\
Ax_2 - \lambda x_2 &= x_1 \\
&\vdots \\
Ax_k - \lambda x_k &= x_{k-1}
\end{aligned}
$$

x_0 is an eigenvector of A corresponding to λ. The vectors $x_1, \ldots, x_k$ are called ***generalized eigenvectors*** of A corresponding to the eigenvalue λ and eigenvector x_0.

Equation 2.1(1) can be written $(A-\lambda I)x_0 = 0, (A - \lambda I)\, x_1 = x_0, \ldots, (A-\lambda I)x_k = x_{k-1}$.

So $(A - \lambda I)x_0 = 0$, $(A - \lambda I)^2 x_1 = 0$, $(A - \lambda I)^3 x_2 = 0$, $\ldots$, $(A - \lambda I)^{k+1} x_k = 0$. Thus we way calculate a Jordan chain into the form $(A - \lambda I)^k x_k$, $(A - \lambda I)^{k-1} x_k$, $\ldots$, $(A - \lambda I)\, x_k$, x_k.

2.2 Definition. The subspace Ker $(A - \lambda I)^p$, integer $p \geq 1$ is called the ***generalized eigenspace*** of A corresponding to eigenvalue λ of A if Ker $(A - \lambda I)^i$= Ker $(A - \lambda I)^p$ for all integer $i>p$ and is denoted by $R_\lambda(A)$. So $R_\lambda(A)$ = Ker $(A - \lambda I)^p$ is the biggest subspace in (1). Since $p \leq n$ we also have $R_\lambda(A) = \{x \in C^n \mid (A - \lambda I)^n x = 0\}$ = Ker $(A - \lambda I)^n$.

2.3 Proposition. The generalized eigenspace $R_\lambda(A)$ contains the vectors from any Jordan chain of A corresponding to λ and $R_\lambda(A)$ is A-invariant.

Proof. Let $x_0, \ldots, x_k$ be a Jordan chain of A corresponding to λ. Then

$$\begin{aligned}(A - \lambda I)^{k+1} x_k &= (A - \lambda I)^k (A - \lambda I)\, x_k \\ &= (A - \lambda I)^k x_{k-1} = (A - \lambda I)^{k-1} x_{k-2} \\ &\;\vdots \\ &= (A - \lambda I)\, x_0 \\ &= 0.\end{aligned}$$

Hence $x_i \in R_\lambda(A)$, $i = 0, \ldots, k$.

If $x \in$ Ker $(A - \lambda I)^n$, then $(A - \lambda I)^n x = 0$.

Thus $(A - \lambda I)^n (Ax) = A((A - \lambda I)^n x) = A0 = 0$.

Hence $R_\lambda(A)$ = Ker $(A - \lambda I)^n$ is A-invariant.

2.4 Lemma. For any eigenvalue λ of A, then (the restriction linear operator of A on $R_\lambda(A)$), $A|_{R_\lambda(A)}$ has only one eigenvalue λ.

Proof. Let λ' be eigenvalue of $A|_{R_\lambda(A)}$.

Then there exists nonzero eigenvector $x \in R_\lambda(A)$ such that $Ax = \lambda' x$. Then

$(A - \lambda I)\, x = \lambda' x - \lambda x = (\lambda' - \lambda)\, x$

$(A - \lambda I)^2 x = (A - \lambda I)\, (\lambda' - \lambda) x = (\lambda' - \lambda)(\lambda' - \lambda) x = (\lambda' - \lambda)^2 x$ and so on, thus we have $(A - \lambda I)^k x = (\lambda' - \lambda)^k x$ for each positive integer k. Since x is generalized eigenvector of A corresponding λ, for some l, then $(\lambda' - \lambda)^l = 0$, thus we have $\lambda' = \lambda$.

2.5 Lemma. If $A : C^n \to C^n$ be a linear operator, then non zero generalized eigenvectors corresponding to distinct eigenvalues of A are linearly independent.

Proof. Suppose $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of A and $v_1, \ldots, v_m$ are corresponding non zero generalized eigenvectors. Suppose

(1) $a_1 v_1 + \ldots + a_m v_m = 0$ for some scalars $a_1, \ldots, a_m$.

Let k be the largest non negative integer such that $(A - \lambda_1 I)^k v_1 \neq 0$ and $(A - \lambda_1 I)^k v_1 = w$. Thus $(A - \lambda_1 I)\, w = (A - \lambda_1 I)^{k+1} v_1 = 0$ and hence $Aw = \lambda_1 w$. Thus $(A - \lambda I)\, w = \lambda_1 w - \lambda w = (\lambda_1 - \lambda) w$, $\forall \lambda \in C$. So $(A - \lambda I)^n w = (\lambda_1 - \lambda)^n w$, $\forall \lambda \in C$, where $n = \dim C^n$. Apply the linear operator

$(A - \lambda_1 I)^k (A - \lambda_2 I)^n \ldots (A - \lambda_m I)^n$ to (1)

$$(A - \lambda_1 I)^k (A - \lambda_2 I)^n \ldots (A - \lambda_m I)^n (a_1 v_1 + \ldots + a_m v_m) = 0$$

$$a_1 (A - \lambda_1 I)^k (A - \lambda_2 I)^n \ldots (A - \lambda_m I)^n v_1 = 0$$

$$a_1 (A - \lambda_2 I)^n \ldots (A - \lambda_m I)^n w = 0$$

$$a_1 (\lambda_1 - \lambda_2)^n \ldots (\lambda_1 - \lambda_m)^n w = 0.$$

This implies that $a_1 = 0$. In a similar fashion $a_j = 0$ for each j. Thus $v_1, \ldots, v_m$ are linearly independent.

2.6 Lemma. Given a linear operator $A : C^n \to C^n$ with an eigenvalue λ, let q be a positive integer for which $\mathrm{Ker}\,(A - \lambda \mathrm{I})^q = R_\lambda(R)$. Then the subspace $\mathrm{Ker}\,(A - \lambda I)^q$ and $\mathrm{Im}\,(A - \lambda I)^q$ are direct complements to each other in C^n.

Proof. Since $\dim \mathrm{Ker}\,(A - \lambda I)^q + \dim \mathrm{Im}\,(A - \lambda I)^q = n$, we have only to check that $\mathrm{Ker}\,(A - \lambda I)^q \cap \mathrm{Im}\,(A - \lambda I)^q = \{0\}$.

For a contradiction, assume that $x \in \mathrm{Ker}\,(A - \lambda I)^q \cap \mathrm{Im}\,(A - \lambda I)^q$, $x \neq 0$.

Then $x = (A - \lambda I)^q y$, for some y and $(A - \lambda I)^r x = 0$ and $(A - \lambda I)^{r-1} x \neq 0$ for some integer $r \geq 1$. Thus $(A - \lambda I)^{q+r} y = 0$ and $(A - \lambda I)^{q+r-1} y \neq 0$. So Ker

$(A - \lambda I)^{q+r} \neq$ Ker $(A - \lambda I)^{q+r-1}$. This contradicts to definition of generalized eigenspace.

2.7 Theorem. Let $\lambda_1, \ldots, \lambda_r$ be all the different eigenvalues of a linear operator $A : C^n \to C^n$. Then C^n decomposes into the direct sum $C^n = R_{\lambda_1}(A) + \ldots + R_{\lambda_r}(A)$.

Proof. For $n = 1$. let λ be an eigenvalue of A, then there exists $v \neq 0$ in C^n such that $Av = \lambda v$. Since $\{v\}$ is a basic of C^n, for each $x \in C^n$ $(A - \lambda I)x = (A - \lambda I)\mu v$ for some $\mu \in C$. So we have $(A - \lambda I)x = \mu\lambda v - \lambda\mu v = 0$. Then $x \in R_\lambda(A)$. Thus $C^n = R_\lambda(A)$.

Let $n > 1$. Assume that the result holds for dimensions $k = 1, 2, \ldots, n - 1$.

Consider the eigenvalue λ_1.

$C^n =$ Ker $(A - \lambda_1 I)^n$ + Im $(A - \lambda_1 I)^n = R_{\lambda_1}(A) + U$. We know that Im $(A - \lambda_1 I)^n = U$ is A-invariant. Since $R_{\lambda_1}(A) \neq 0$, we have dim $U < n$. By Proposition 2.3, there does not exist generalized eigenvectors of $A|_U$ corresponding to the eigenvalue λ_1. Thus each eigenvalue of $A|_U$ corresponding to the eigenvalue λ_1. Thus each eigenvalue of $A|_U$ is in $\{\lambda_2, \ldots, \lambda_r\}$. By induction hypothesis $U = R_{\lambda_2}(A|_U) + \cdots + R_{\lambda_r}(A|_U)$. Thus $C^n = R_{\lambda_1}(A) + R_{\lambda_2}(A|_U) + \cdots + R_{\lambda_r}(A|_U)$. So we show that $R_{\lambda_k}(A) = R_{\lambda_k}(A|_U)$ for $k = 2, \ldots, m$. Take a fixed integer $k \in \{2, \ldots, m\}$ and clearly $R_{\lambda_k}(A|_U) \subseteq R_{\lambda_k}(A)$. Assume $R_{\lambda_k}(A|_U) \neq R_{\lambda_k}(A)$. Then there exists $v \in R_{\lambda_k}(A)$ but $v \notin R_{\lambda_k}(A|_U)$. So we get $v \in R_{\lambda_j}(A|_U)$ for some $j \neq k$ and hence $v \in R_{\lambda_j}(A)$. Thus $v \in R_{\lambda_k}(A) \cap R_{\lambda_j}(A)$. This contradicts to lemma 2.5. So $R_{\lambda_k}(A) = R_{\lambda_k}(A|_U)$. Thus $C^n = R_{\lambda_1}(A) + \cdots + R_{\lambda_r}(A)$.

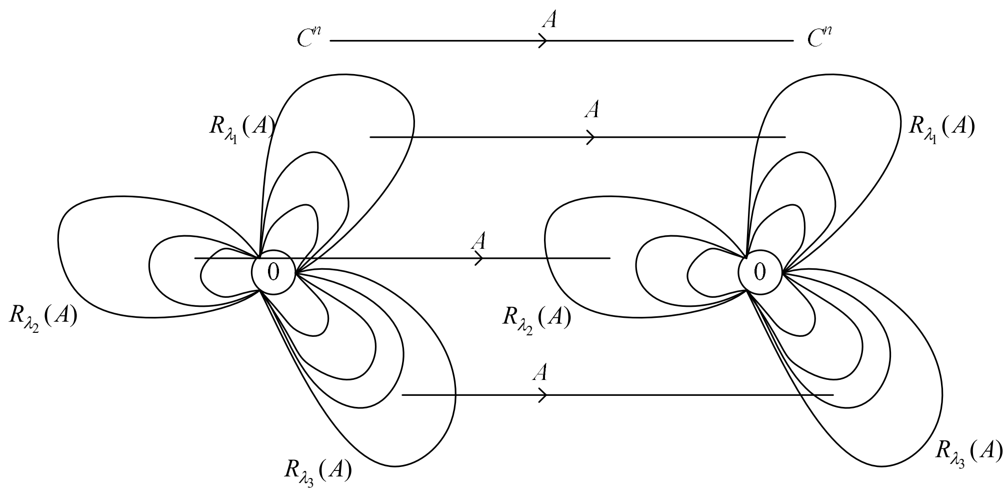

Figure 1

3. Jordan Subspaces

3.1 Definition. An A-invariant subspace M is called a ***Jordan subspace*** corresponding the eigenvalue λ_0 of A if M is spanned by the vectors of some Jordan chain of A corresponding to λ_0.

3.2 Proposition. Let $A : C^n \to C^n$ be a linear operator. Let $x_0, x_1, \ldots, x_k$ be a Jordan chain of a linear operator A corresponding to λ_0. Then the subspace $M = \text{Span}\ \{x_0, \ldots, x_k\}$ is A-variant.

Proof. We have $Ax_0 = \lambda_0 x_0 \in M$ where λ_0 is the eigenvalue of A and for $i = 1, \ldots, k$, $Ax_i = \lambda_0 x_i + x_{i-1} \in M$. Hence M is A-invariant.

3.3 Theorem. Let $A : C^n \to C^n$ be a linear operator. Then there exists a direct sum decomposition

(1) $$C^n = M_1 + \ldots + M_p$$

where M_i is a Jordan subspace of A corresponding to an eigenvalue λ_i ($\lambda_1, \ldots, \lambda_p$ are not necessarily different).

Proof. Assume A has only one eigenvalue λ_0, (possibly with there are more one eigenvalue, all equal to λ_0).

Let $Y_j = \text{Ker } (A - \lambda_0 I)^j$, $j = 1, 2, \ldots, m$, where m is chosen $Y_m = R_{\lambda_0}(A)$ and $Y_{m-1} \neq R_{\lambda_0}(A)$. So $Y_1 \subset Y_2 \subset \ldots \subset Y_m$. Let $x_m^{(1)}, \ldots, x_m^{(t_m)}$ is a basis of Y_m modulo Y_{m-1}. So $x_m^{(1)}, \ldots, x_m^{(t_m)}$ are linearly independent in set Y_m such that

(2) $$Y_{m-1} + \text{Span } \{x_m^{(1)}, \ldots, x_m^{(t_m)}\} = Y_m \text{ (the sum is here direct)}$$

Claim that the mt_m vectors $(A - \lambda_0 I)^k x_m^{(1)}, \ldots, (A - \lambda_0 I)^k x_m^{(t_m)}$, $k = 0, \ldots, m-1$ are linearly independent. Let

(3) $$\sum_{k=0}^{m-1} \sum_{i=1}^{t_m} \alpha_{ik} (A - \lambda_0 I)^k x_m^{(i)} = 0, \qquad \alpha_{ik} \in C .$$

Apply $(A - \lambda_0 I)^{m-1}$ and use the property $(A - \lambda_0 I)^m x_m^{(i)} = 0$, for $i = 1, \ldots, t_m$.

Thus $(A - \lambda_0 I)^{m-1} \left\{ \sum_{i=1}^{t_m} \alpha_{i0} x_m^{(i)} \right\} = 0$. So $\sum_{i=1}^{t_m} \alpha_{i0} x_m^{(i)} \in Y_{m-1}$.

By 3.3(2), $\sum_{i=1}^{t_m} \alpha_{i0} x_m^{(i)} \in Y_{m-1} \cap \text{Span } \{x_m^{(1)}, \ldots, x_m^{t_m}\}$ and so $\alpha_{10} = \cdots = \alpha_{t_m 0} = 0$.

Apply $(A - \lambda_0 I)^{m-2}$ to 3.3(3) we show similarly that $\alpha_{11} = \cdots = \alpha_{t_m 1} = 0$ and so on.

We put $$M_1 = \text{Span } \{(A - \lambda_0 I)^k x_m^{(1)}, k = 0, \ldots, m-1\}$$

$$M_2 = \text{Span } \{(A - \lambda_0 I)^k x_m^{(2)}, k = 0, \ldots, m-1\}$$

$$\vdots$$

$$M_{t_m} = \text{Span } \{(A - \lambda_0 I)^k x_m^{(t_m)}, k = 0, \ldots, m-1\}.$$

Since $M_i \cap M_j = \{0\}$ for $i \neq j$, then the sum $M_1 + M_2 + \ldots + M_{t_m}$ is direct. Now consider the linear independent vectors $x_{m-1}^{(i)} = (A - \lambda_0 I)x_m^{(i)}$, $i = 1, \ldots, t_m$. Claim that

(4) $$Y_{m-2} \cap \text{Span}\, \{x_{m-1}^{(1)}, x_{m-1}^{(2)}, \ldots, x_{m-1}^{(t_m)}\} = \{0\}.$$

Let $\sum_{i=1}^{t_m} \alpha_i x_{m-1}^{(i)} \in Y_{m-2}$, $\alpha_i \in C$. Apply $(A - \lambda_0 I)^{m-2}$ to the left-hand side, we get

$(A - \lambda_0 I)^{m-2} \sum_{i=1}^{t_m} \alpha_i (A - \lambda_0 I)x_m^{(i)} = 0$, So $(A - \lambda_0 I)^{m-1} \sum_{i=1}^{t_m} \alpha_i x_m^{(i)} = 0$, which implies $\alpha_1 = \cdots = \alpha_{t_m} = 0$. So the equation 3.3(4) follows. Assume that $Y_{m-2} + \text{Span}\, \{x_{m-1}^{(1)}, \ldots, x_{m-1}^{(t_m)}\} \neq Y_{m-1}$. Then there exist vectors $x_{m-1}^{(t_m+1)}, \ldots, x_{m-1}^{(t_m+t_{m-1})} \in Y_{m-1}$ such that $\{x_{m-1}^{(i)}\}_{i=1}^{t_m+t_{m-1}}$ is linearly independent and

(5) $$Y_{m-2} + \text{Span}\, \{x_{m-1}^{(1)}, \ldots, x_{m-1}^{(t_m+t_{m-1})}\} = Y_{m-1}.$$

Applying previous argument to 3.3(5) as with 3.3(2), we fine that the vectors $(A - \lambda_0 I)^k x_{m-1}^{(1)}, \ldots, (A - \lambda_0 I)^k x_{m-1}^{(t_m+t_{m-1})}, k = 0, \ldots, m-2$ are linearly independent.

Now let $$M_{t_m+1} = \text{Span}\, \{(A - \lambda_0 I)^k x_{m-1}^{(t_m+1)},\, k = 0, \ldots, m-2\}$$

$$\vdots$$

$$M_{t_m+t_{m-1}} = \text{Span}\, \{(A - \lambda_0 I)^k x_{m-1}^{(t_m+t_{m-1})},\, k = 0, \ldots, m-2\}.$$

If $Y_{m-2} + \text{Span}\{x_{m-1}^{(1)}, \ldots, x_{m-1}^{(t_m)}\} = Y_{m-1}$, then $t_{m-1} = 0$.

At the next step put $x_{m-2}^{(i)} = (A - \lambda_0 I)x_{m-1}^{(i)}$, $i = 1, \ldots, t_m + t_{m-1}$ and show similarly that $Y_{m-3} \cap \text{Span}\{x_{m-2}^{(i)}, i = 1, \ldots, t_m + t_{m-1}\} = \{0\}$.

Assume that that $Y_{m-3} \cap \text{Span}\{x_{m-2}^{(i)}, i=1,\ldots,t_m+t_{m-1}\} \neq Y_{m-2}$, then there exist vectors $x_{m-2}^{(i)} \in Y_{m-2}$, $i = t_m + t_{m-1} + 1, \ldots, t_m + t_{m-1} + t_{m-2}$ such that $x_{m-2}^{(i)}$, $i = 1, \ldots, t_m + t_{m-1} + t_{m-2}$ are linearly independent and $Y_{m-3} + \text{Span}\ \{x_{m-2}^{(i)}, i = 1, \ldots, t_m + t_{m-1} + t_{m-2}\} = Y_{m-2}$.

We continue this process of construction of M_i, $i = 1, \ldots, p$ where $p = t_m + t_{m-1} + \ldots + t_1$.

The construction shows that each M_i is Jordan subspace of A and $M_1 + \ldots + M_p$ is a direct sum. Also $M_1 + \ldots + M_p = R_{\lambda_0}(A) = C^n$.

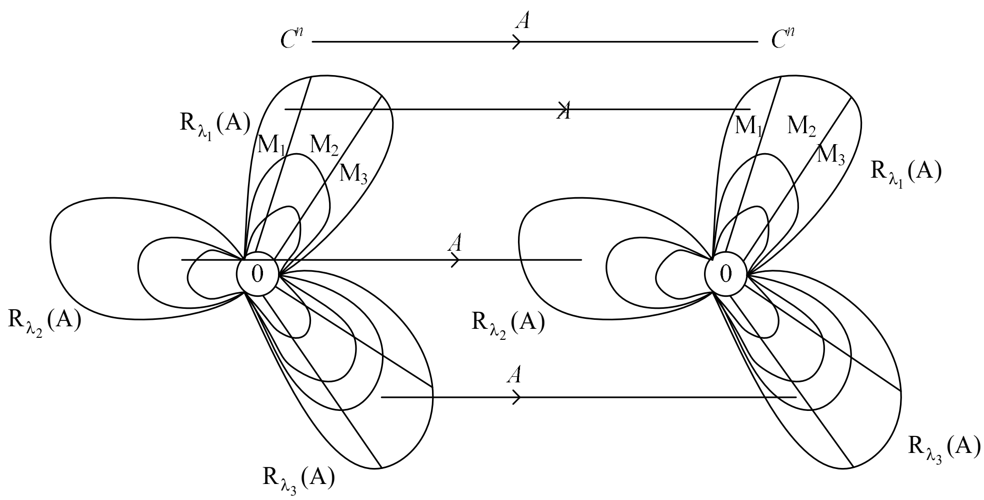

Figure. 2

3.4 Example. Let us consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$|A - \lambda I| = (2-\lambda)^5$

λ= 2, 2, 2, 2, 2 are eigenvalues of A.

$$A-2I=\begin{bmatrix}0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\end{bmatrix},(A-2I)^2=\begin{bmatrix}0&0&1&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix},(A-2I)^2=\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$

$Y_1 = Ker(A-2I) = \{\lambda_1 e_1 + \lambda_2 e_4 \mid \lambda_1, \lambda_2 \text{ are scalars}\}$

$Y_2 = Ker(A-2I)^2 = \{\mu_1 e_1 + \mu_2 e_2 + \mu_3 e_4 + \mu_4 e_5 \mid \mu_1, \mu_2, \mu_3, \mu_4 \text{ are scalars}\}$

$Y_3 = Ker(A-2I)^3 = \{\eta_1 e_1 + \eta_2 e_2 + \eta_3 e_3 + \eta_4 e_4 + \eta_5 e_5 \mid \eta_1, \eta_2, \eta_3 + \eta_4 + \eta_5 \text{ are scalars}\}$

$$Y_1 \subset Y_2 \subset Y_3 = R_2(A) = C^5$$

e_3 is a basis of Y_3 modulo Y_2 such that

$$Y_2 + \text{span } \{e_3\} = Y_3$$

Jordan subspace $M_1 = \text{Span}\{A-2I)^2 e_3, (A-2I)e_3, e_3\} = Span\{e_1, e_2, e_3\}$

$$(A-2I)e_3 = e_2 \in Y_2$$

$$Y_1 + \text{Span}\{(A-2I)e_3\} \neq Y_2$$

$\exists e_5 \in Y_2$ such that $\{e_2, e_5\}$ is linearly independent set.

Jordan subspace M_2= Span$\{(A-2I)e_5, e_5\} = Span\{e_4, e_5\}$

$$M_1 + M_2 = R_2(A) = C^5 = Y_3$$

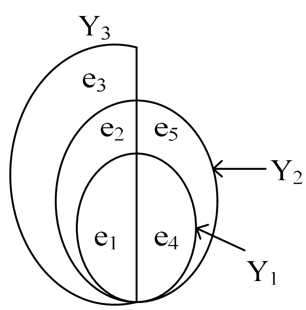

Figure. 3

References

1. S. Axler, Down with Determinants!, *American Mathematical Monthly* **102**(1995) 139-154.
2. S. Axler, *Linear Algebra Done Right*, Third Edition, Springer International Publishing, 2015.
3. I. Gohberg, P.Lancaster & L. Rodman, *Invariant Subspaces of Matrices with Application*, John Wiley & Sons, New York, 2006.
4. P. R. Halmos, *Finite-Dimensional Vector Spaces*, Second Edition, Springer-Verlag, New York, 1987.